2 – Numerical solution of Volterra integral equations of the second kind with logarithmic kernels by orthogonal polynomials

الحل العددي لمعادلات فولتيرا التكاملية من النوع الثاني ذات النوى اللوغاريتمية باستخدام كثيرات الحدود المتعامدة

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مستخلص

تُستخدم المعادلات التكاملية ذات النوى الضعيفة الشاذة في العديد من مسائل الفيزياء الرياضية والميكانيكا التطبيقية وفيزياء الكم ودراسة التفاعلات الكيميائية والكيمياء الكهربائية وغيرها. ويُعدّ إيجاد الحلول العددية لهذه الفئة من المعادلات ولا سيما صيغتها غير الخطية أمرًا بالغ التعقيد. وتُعتبر النوى اللوغاريتمية من أبرز خصائص هذه الفئة من المعادلات والتي استقطبت اهتمام العديد من الباحثين في السنوات الأخيرة. استندت هذه الورقة البحثية إلى الحل العددي لمعادلات فولتيرا التكاملية الضعيفة الشاذة من النوع الثاني ذات النوى اللوغاريتمية باستخدام كثيرات الحدود المتعامدة. ويهدف هذا البحث إلى دراسة الحل العددي لهذا النوع من المعادلات باستخدام طريقة التوسع المحدود

لكثيرات حدود تشيبيشيف وليجندر، بالإضافة إلى تحليل الخطأ والتقارب أخيرًا، تمّت دراسة النتائج العددية وتقارب هاتين الطريقتين ومقارنتها ببعض الطرق العددية الأخرى بما في ذلك طريقة التموضع المشترك باستخدام كثيرات حدود برونر المتصلة القطعية لهذا النوع من المعادلات وأظهرت النتائج التي تمّ الحصول عليها من هذا البحث أنه حقق كفاءةً ونتائج أفضل من الأبحاث المماثلة الأخرى

Abstract

Integral equations with weak singular kernels are used in many problems of mathematical physics, applied mechanics, quantum physics, study of chemical reactions, electrochemistry, etc. Determining numerical solutions of this class of equations, especially its nonlinear form, has many complexities. Logarithmic kernels are among the prominent characteristics of this class of equations, which have attracted the attention of many researchers in recent years. This paper was based on the numerical solution of weak singular Volterra integral equations of the second kind with logarithmic kernels using orthogonal polynomials. The purpose of this research is to investigate the numerical solution of this type of equations using the finite expansion method of Chebyshev and Legendre polynomials, as well as their error and convergence analysis. Finally, the numerical results and convergence of these two methods were investigated and compared with some numerical methods, including the co-localization method with continuous piecewise Brunner polynomials for this type of equations. The results obtained from this research showed that it showed more efficient and better results than other similar research.

Keywords: Integra-Volterra equations – Legendre polynomials – Chebyshev polynomials – logarithmic kernels.

مقدمة

تُستخدم المعادلات التكاملية بكثرة في الفيزياء الرياضية، مثل ديناميكا الموائع وفيزياء البلازما. وتحدث هذه الظاهرة عادةً عندما تعتمد ظاهرة أو كمية ما في زمان ومكان معينين على جميع الأزمنة (الأماكن) السابقة. وتؤدي هذه المسألة إلى معادلة تكاملية. في هذه الحالة، تربط معادلة بواسون الجهد بكثافة الشحنة [١–2]. تشمل التطبيقات الأخرى للمعادلات التكاملية الجهد الكهربائي على حافة قرص واحد، ومسألة عدد السكان، ومسألة تآكل المعدات ومعدل استبدالها، والتواء الأسلاك، ومسألة التحكم الآلي وانحراف عمود دوار، والعلوم التطبيقية مثل

الميكانيكا، وعلوم الاتصالات، والبتروكيماويات، وبناء المباني والجسور، والليزر، ومحطات الطاقة النووية، والمفاعلات، والنظرية، والموجات الإلكترونية، والمعادلات التفاضلية العادية، والمعادلات التفاضلية الجزئية [٣]. من منظور تاريخي، يعود ظهور المعادلات التكاملية إلى عام ١٧٨٢، ولذلك تُشير العديد من المصادر الرياضية والتاريخية إلى لابلاس باعتباره مؤسس هذا الفرع من الرياضيات. في بعض المصادر، يُعتبر الانتقال من القرن التاسع عشر إلى القرن العشرين بدايةً لتطور المعادلات التكاملية. وقد لعب علماءٌ مثل فوريه (1811 و1822)، وآبل وبواسون (1826)، وليوفيل (1832)، ثم نيومان (1870)، وبوانكاريه وفولتيرا (1896)، وفريدهيلم (1900–1903) دورًا بارزًا في هذا المجال. كما أجرى هيلبرت أبحاثًا في المعادلات التكاملية، وصاغ مسائل المعادلات التفاضلية في صورة معادلات تكاملية. وكان أول من قسّم المعادلات التكاملية إلى نوعين: النوع الأول والنوع الثاني، واكتشف طريقةً لحل هذه المعادلات باستخدام متسلسلات القوى. ومن أهم إنجازات هيلبرت في عامي 1904 و1905 صياغة مسألة ستورم–ليوفيل للقيم الحدية في صورة معادلة تكاملية. منذ عام 1985، دأب علماء الرياضيات المعاصرون، مثل بلينز و ر. كريس وك. أتكينسون، على تطوير طرق عددية لحل المعادلات التكاملية، وإثبات نظريات تفرد واستمرارية الحلول، والاستقرار العددي وتقارب هذه الطرق. يظهر نوع خاص من معادلات فولتيرا التكاملية ذات النوى الشاذة الضعيفة في العديد من المسائل، بما في ذلك النمذجة في الفيزياء الرياضية، وانتقال الحرارة، والكيمياء الكهربائية، والإشعاع الحراري من المواد الصلبة [4]، والعديد من المسائل التطبيقية الأخرى. في السنوات الأخيرة، تم تناول هذا النوع من المعادلات في العديد من المقالات

1-Introduction

Integral equations are often used in mathematical physics, such as fluid dynamics and plasma physics. This phenomenon typically occurs when a phenomenon or quantity at some time and place depends on all previous times (places). This problem leads to an integral equation. In this case, Poisson's equation relates the potential to the charge density [1-2]. Other applications of integral equations include the electric potential on the edge of a single disk, the human population problem, the problem of equipment wear and tear and its replacement rate, the twisting of a wire, the problem of automatic control and deflection of a rotating shaft, applied sciences such as mechanics, communication sciences, petrochemicals, building and bridge construction, lasers, nuclear power plants, reactors, theory, electronic waves, ordinary differential equations and partial differential equations [3]. From a historical perspective, the emergence of integral equations dates back to 1782, so that many mathematical and mathematical history sources introduce Laplace as the founder of this branch of mathematics. In some sources, the beginning of the development of integral equations is the transition period from the 19th to the 20th century. People such as Fourier in 1811 and 1822, Abel and Poisson in 1826, Liouville in 1832, then Neumann in 1870, Poincaré and Volterra in 1896, and Fredhelm in 1900 to 1903, who played a significant role in the field of integral equations. Hilbert also conducted research on integral equations, and set the problems of differential equations as an integral equation. He was the first to divide integral equations into two classes of the first and second types, and was able to discover a method based on power series to solve these equations. One of Hilbert's most important results in 1904 and 1905 was the formulation of the Sturm-Liouville boundary value problem as an integral equation. Since 1985, contemporary mathematicians such as Plains, R. Kress, and K. Atkinson have been compiling numerical methods for solving integral equations, proving the theorems of uniqueness and continuity of solutions, and numerical stability and convergence of these methods. A special type of Volterra integral equations with weak singular kernels arise in many problems, modeling in mathematical physics, heat transfer, electrochemistry, heat radiation from solids [4] and many other applied problems. In recent years, this type of equations has been considered in many articles, including [5-6].

Consider the following integral equation:

$$y(t) = g(t) + \int_0^t ln(t-s)\big(t,s,y(s)\big)ds,$$

Where the function $f(t,s,y\ (s))$، times $\nu + 1$is the ratio $(\nu \geq 1)$to the variables t, s and $y(s)$where:

$$(t \in I = [0,T], s \in [0,t, y(s) \in R)$$

And also the function $g(t)$, $\nu + 1$ times $(\nu \geq 1)$, with respect to the variable $\epsilon(0,T)$ to t are continuously differentiable. In general, suppose that there is a unique and continuous solution on the interval I. The existence and uniqueness of this solution under favorable conditions have been investigated in [6-7]. Since in general this solution cannot be obtained by analytical methods, therefore it is important to find an exact solution using numerical methods. In this chapter we describe and investigate a numerical method using Chebyshev and Legendre polynomials, which was proposed by Khater et al. in [1-2]. Suppose the partition ∇ on the interval $I = [0, T]$, with step length $h = t_{i+1} - t_i$ is as follows:

$$\nabla = \{0 = t_0 < t_1 < \cdots < T_N = T\}. \qquad (1)$$

Each subinterval ($[t_i, t_{i+1}]$)is divided by the points $t_i + C_j hi$ where $C_j$is defined as follows:

$$C_j = \frac{1}{2}\left(1 - \cos\left(\frac{j\pi}{\nu}\right)\right), \qquad j = 0,1, \dots, \nu, \qquad (2)$$

Where $C_j$are the Gaussian Chebyshev-Lebato points shifted to the interval [0,1]. Also, the unique choice of $C_0$and $C_\nu$in (2) has led to the continuity of the approximate solution on the interval $[0, T]$ [8-9.]

2- Numerical solution using Chebyshev polynomials

This method uses the finite expansion of Chebyshev polynomials to approximate the solution of the integral equation on the interval ($i = t \in [t_i, t_{i+1}], 0,1, N - 1$):

$$y(t) \approx y_{i+1}(t) = \sum_{r=0}^{\nu\;''} a_r^{(i+1)}\; T_r\left[\frac{2}{h_i}(t - t_i) - 1\right], \qquad (3)$$

So that $T_r[t]$is the rth Chebyshev polynomial. The symbol $''$in the upper bound of sigma represents half the sum of the first and last two terms of sigma [10]. To obtain the coefficients of $a_r^{(i+1)}$we multiply equation (3) by the following expression:

$$\frac{T_k\left[\frac{2}{h_i}(t - t_i) - 1\right]}{\sqrt{1 - \left(\frac{2}{h_i}(t - t_i) - 1\right)^2}}$$

Then we integrate both sides of the relation over the interval $[t_i, t_{i+1}]$, Thus, the following expression is obtained:

$$\int_{t_i}^{t_{i+1}} \frac{y_{i+1}(t) T_K\left[\frac{2}{h_i}(t - t_i) - 1\right]}{\sqrt{1 - \left(\frac{2}{h_i}(t - t_i) - 1\right)^2}} dt$$

$$= \sum_{r=0}^{\nu\;''} \int_{t_i}^{t_{i+1}} \frac{T_r\left[\frac{2}{h_i}(t - t_i) - 1\right] T_k\left[\frac{2}{h_i}(t - t_i) - 1\right]}{\sqrt{1 - \left(\frac{2}{h_i}(t - t_i) - 1\right)^2}} dt$$

$$k = \int_0^{1-\xi} x(x+\xi-1)\ln\left(\frac{h_i}{2}x\right)T_r[x+\xi]dx$$

$$\int_0^{1-\xi} [x+\xi-\xi]\,[(x+\xi)-1]\ln\left(\frac{h_i}{2}x\right)T_r[x+\xi]dx$$

$$\int_0^{1-\xi} x(x+\xi)^2\ln\left(\frac{h_i}{2}x\right)T_r[x+\xi]dx$$

$$-(1+\xi)\int_0^{1-\xi} ln\left(\frac{h_i}{2}x\right)(x+\xi)\,T_r[x+\xi]dx$$

$$+\int_0^{1-\xi} ln\left(\frac{h_i}{2}x\right)T_r[x+\xi]dx \qquad (8)$$

Using the two equations (9) and (10), equation (11) is obtained as follows:

$$xT_r[x] = \frac{1}{2}(T_{r+1}[x] + T_{r+1}[x]), \qquad (9)$$

$$x^2T_r[x] = \frac{1}{4}(T_{r+2}[x] + 2T_r[x] + T_{r-2}[x]), \qquad (10)$$

$$K = \int_0^{1-\xi} \ln\left(\frac{h_i}{2}x\right)\left(\frac{1}{4}(T_{r+2}[x+\xi] + 2T_r[x+\xi] + T_{r-2}[x+\xi])\right)dx$$

$$-(1-\xi)\int_0^{1-\xi} \ln\left(\frac{h_i}{2}x\right)\frac{1}{2}(T_{r+1}[x+\xi] + T_{r-1}[x+\xi])dx$$

$$\xi\int_0^{1-\xi} \ln\left(\frac{h_i}{2}x\right)T_r[x+\xi]dx.\,[x+\xi] \qquad (11)$$

Considering equations (7) and (11), we have:

$$K = \frac{1}{4}\Big(I_{r+2j}(\xi) + 2I_{r-2}(\xi)\Big) - \frac{1+\xi}{2}\Big(I_{r+1j}(\xi) + I_{r+1j}(\xi)\Big) + \xi I_{rj}(\xi). \qquad (12)$$

On the other hand, K is also calculated by integrating equation (8) part by part, which after simplification leads to equation (13) [2]:

$$K = -\frac{1}{4(r+1)}\Big(I_{r+2j}(\xi) + I_{rj}(\xi)\Big) + \frac{1}{2(r+1)}I_{r+1j}(\xi)$$

$$+\frac{1}{4(r-1)}\Big(I_{r-2j}(\xi) + I_{rj}(\xi)\Big) - \frac{1}{2(r+1)}I_{r-1j}(\xi)$$

$$-\frac{1}{4(r+1)}\Big(I_{r+2j}(\xi) + I_{rj}(\xi)\Big) + \frac{\xi}{2(r+1)}I_{r+1j}(\xi)$$

$$\frac{1}{4(r-1)}\Big(I_{rj}(\xi) + I_{r,j-2}(\xi)\Big) - \frac{\xi}{2(r-1)}I_{r-1j}(\xi)$$

$$-\frac{1}{4(r+1)}\left(M_{r+2j}(\xi)+M_{rj}(\xi)\right)+\frac{\xi 1}{2(r+1)}M_{r+1j}(\xi)$$

$$\frac{1}{4(r-1)}\left(M_{r+2j}(\xi)+M_{r-2j}(\xi)\right)-\frac{\xi}{2(r-1)}M_{r-1j}(\xi) \qquad (14)$$

Where $M_{rj}(\xi)$ is obtained using the following two equations:

$$M_{rj}(\xi)=\int_0^{1-\xi}T_r[x+\xi]dx, \qquad M_{1,j}(\xi)=\frac{1-\xi^2}{2}$$

$$2(r^2-1)M_{r,j}(\xi)$$
$$=-2-(r-1)T_{r+1}[\xi]+(r+1)T_{r-1}[\xi], \quad (r=2,3,\dots,\nu)$$

Finally, by equating equations (13) and (12) and then multiplying both sides by $4(r^2-1)$, the recursive equation for obtaining $I_{rj}(\xi)$is obtained as follows:

$$(r-1)(r+3)I_{r+2,j}(\xi)-2(1+\xi)(r-1)(r+2)I_{r+2,j}(\xi)$$
$$+2[(r^2-1)(1+2\xi)-2]I_{rj}(\xi)$$
$$-2(1+\xi)(r+1)(r-2)I_{r+2,j}+(r+1)(r-3)I_{r+2,j}(\xi)=(1-$$
$$r)M_{r+2,j}+2(r-1)M_{r+1,j}(\xi)+2(r-1)M_{r+2,j}(\xi)+2M_{r+2,j}(\xi)-$$
$$2(r+1)M_{r+2,j}+(r+1)M_{r-2,j}(\xi), \quad (r=2,3,\dots,\nu-2, j=$$
$$0,1,\dots,\nu) \qquad (14)$$

To increase the speed of the integral (7), the recursive relation (14) is used. The initial values for this recursive relation are calculated directly using the part-by-part integration method as follows:

$$I_{0,j}(\xi)=2C_j\{ln(h_jC_j)-1\},$$

$$I_{1,j}(\xi)=\frac{1}{2}C_j\{1+\xi[2\,ln(h_jC_j)-1]-2\xi\}$$

$$I_{2,j}(\xi)=\frac{2}{9}C_j\{(2\xi^2+2\xi-1)[3\,ln(h_jC_j)-5]-(\xi^2-5\xi-2)\}$$

$$I_{3,j}(\xi)=\frac{1}{6}C_j\{6(1+\xi)(2\xi^2-1)\,ln(h_jC_j)-(13\xi^2-6)(2\xi+1)$$
$$+\xi(8+\xi^2)\}$$

By inserting the expression $Ir,j\ (cos\,\frac{j\pi}{\nu})$ into (6) and also using (4) and then rearranging it, the following relation is obtained:

$$\left[\int_{ti}^{tij}ln(t_{ij}-t)y_{j+1}(t)dt\right]=h_jB[y_j+1,j] \qquad (15)$$

Where $y_{i+1,j}(t_{ij})$ is also the elements of matrix Bas follows:

$$b_{jl} = \frac{1}{\nu(1+\delta_{l0}+\delta_{l\nu})}\sum_{r=0}^{\nu''}(-1)^r T_r\left[-\cos\left(\frac{l\pi}{\nu}\right)\right] I_{r,j}\left(\cos\left(\frac{j\pi}{\nu}\right)\right),\ j = 0,1 \dots . \nu,\ \ l = 0,1,\dots,\nu$$

Where $\delta_{i,j}$is the Kronecker delta. In the following, the integral approximation (15) is used to obtain the numerical solution of the integral equation (1). Considering $t = t_{i,j} \in [t_i, t_{i+1}]$in (1), the following relation is obtained:

$$y(t_{i,j}) = g(t_{ij}) + \int_0^{t_{i,j}} \ln(t_{i,j}-s) f\left(t_{i,j}, s, y(s)\right) ds = g(t_{ij}) + \sum_{l=0}^{i-1}\int_{t_l}^{t_{l+1,j}} \ln(t_{i,j}-s)\, f\left(t_{i,j}, s, y(s)\right) ds + \int_{t_l}^{t_{l+1,j}} ln(t_{i,j} - s)\, f\left(t_{i,j}, s, y(s)\right) ds, \quad i = 0,1, \dots N-1, j = 1,2, . . \nu \qquad (16)$$

Then, by putting (15) into (16), the numerical solution to the integral equation will be as follows:

$$y_{i+1,j} = g(t_{i,j}) + \sum_{l=0}^{i-1} hl \sum_{k=0}^{\nu} C_{\nu k} \ln(t_{i,j} - t_{lk}) f(t_{i,j}, t_{lk}, y_{l+1,k}) + h_i \sum_{k=0}^{\nu} b_{jk}\, f(t_{i,j}, t_{i,k}, y_{i+1,k}) \qquad (17)$$

In which:

$$C_{\nu k} = b_{\nu k} = \frac{1}{\nu(1+\delta_{k0}+\delta_{k\nu})}\left\{1 + \sum_{r=2}^{\nu} \frac{\nu}{1+\delta_{\nu r}} \frac{1+(-1)^r}{1-r^2} T_r\left[-\cos\left(\frac{k\pi}{\nu}\right)\right]\right\}$$

$$y_{0,1} = g(0), \qquad y_{i+1,0} = y_{i,\nu}, \quad i = 0,1,\dots,N-1, j = 1,2,\dots,\nu$$

In equation (17), to calculate ν, the unknown $y_{i+1,0}$, in each subinterval $[t_i, t_{i+1}]$ $(i = 0,1,\dots,N-1)$, the Gauss elimination method is used for linear equations and the Newton iterative method is used for nonlinear equations.

3- Numerical solution using Legendre polynomials

In this method, the finite Legendre expansion is used to approximate the solution of the integral equation on the subinterval $t \in [t_i, t_{i+1}]$ $(i = 0,1,\dots,N-1)$:

$$y(t) \approx y_{i+1}(t) = \sum_{r=0}^{\nu} a_r^{(i+1)} P_r\left[\frac{2}{h_i}(t - t_i) - 1\right] \qquad (18)$$

So that $P_r[t]$ is the Legendre polynomial. To obtain the coefficients of $a_r^{(i+1)}$, we multiply equation (18) by the expression $P_k\left[\frac{2}{h_i}(t-t_i)-1\right]$, then we integrate both sides of the equation over the interval $[t_i, t_{i+1}]$. Therefore, the following expression is obtained:

$$\int_{t_i}^{t_{i+1}} y_{i+1}(t)P_k\left[\frac{2}{h_i}(t-t_i)-1\right]dt$$
$$= \sum_{r=0}^{\nu} a_r^{(i+1)} P_r\left[\frac{2}{h_i}(t-t_i)-1\right] \sum_{r=0}^{\nu} a_r^{(i+1)} P_k\left[\frac{2}{h_i}(t-t_i)\right.$$
$$\left. -1\right] dt$$

Then, by changing the variable $s=\frac{2}{h_i}(t-t_i)-1$, the following expression is obtained:

$$\frac{2}{h_i}\int_{-1}^{1} y_{i+1}\left(\frac{h_i}{2}(s+1)+t_i\right)p_k[s]ds = \frac{h_i}{2}\sum_{r=0}^{\nu} a_r^{(i+1)}\int_{-1}^{1} p_r[s]p_k[s]ds.$$

Also, considering the relationship,

$$\int_{-1}^{1} p_r[s]p_k[s]ds = \begin{cases} \frac{2}{2k+1}, & r=k \\ 0, & r\neq k \end{cases}$$

The following expression is obtained:

$$\int_{-1}^{1} y_{i+1}\left(\frac{h_i}{2}(s+1)+t_i\right)p_k[s]ds = \frac{2}{2k+1}a_k^{(i+1)}$$

According to the integration rule [11], the following expression results:

$$a_k^{(i+1)} = (2k+1)\left[-\cos\left(\frac{l\pi}{\nu}\right)\right] y_{i+1,l}, \quad k=0,1,\dots,\nu \qquad (19)$$

Where $y_{i+1,l}=y_{i+1,l}(t_{il})$and $R_l$are the weights of this integration rule. Also, for this rule to be , the weights are obtained using the following equations:

$$R_0x_0^i + R_1x_1^i + \cdots + R_\nu x_\nu^i = G_i, \quad i=0,1,\dots\nu$$

In which:

$$G_i = \begin{cases} \frac{1}{2^i(i+1)} if \ \ iis\ even \\ 0, if \ \ iis\ odd \end{cases}$$

&

$$x_l = -(1/2)\cos\left({}^{l\pi}/_{\nu}\right)$$

To obtain an approximate solution to the integral equation, we multiply equation (18) by $[t_i, t_{ij}]$ ‹in the subinterval $ln(t_{ij} - t)$, then we integrate with respect to t in the given subinterval:

$$\int_{t_i}^{t_{ij}} ln(t_{ij} - t)\, y_{i+1}(t)dt = \sum_{r=0}^{\nu} a_r^{(i+1)} \int_{t_i}^{t_{ij}} ln(t_{ij} - t)\, P_r\left[\frac{2}{h_i}\left((t_{ij} - t) - 1\right)\right] dt. \quad (20)$$

Equation (20) is transformed into the following equation using the change of variable $t = t_i + \frac{h_i}{2}\left(1 - \cos\left(\frac{j\pi}{\nu}\right) - x\right)$:

$$\int_{t_i}^{t_{ij}} ln(t_{ij} - t)\, y_{i+1}(t)dt = \frac{h_i}{2}\sum_{r=0}^{\nu}(-1)^r a_r^{(i+1)} I_{r,j}\, dt\left(cos\left(\frac{j\pi}{\nu}\right)\right)$$
(21)

Where $I_{r,j}\,(\xi)$is defined as follows:

$$I_{r,j}\,(\xi) = \int_0^{1-\xi} \ln\left(\frac{h_i}{2}x\right) p_r[x + \xi]dx. \qquad (22)$$

The recursion relation to obtain $I_{r,j}\,(\xi)$results as follows:

$$\begin{aligned}
&(2r-1)(r+3)(r+2)I_{r+2}(\xi) - (1+\xi)(2r-1)(2r+3)I_{r+1}(\xi) \\
&+[(2r-1)(r+1)(r+3) + r(2r+3)(r-2) \\
&\qquad + \xi(2r-1)(2r+1)(2r+3)]I_{\,r}(\xi) \\
&-(1+\xi)(2r+3)(2r-1)(r-1)I_{r-1}(\xi) + (2r+3)(r-1)(r-2)I_{r-2}(\xi) \\
&= (1-2r)(r+2)M_{r+2}(\xi) + (2r-1)(2r+3)M_{r+1} + (2r+1)M_r(\xi) \\
&\qquad -(2r-1)(2r+3)M_{r-1}(\xi) + \big(r - 1(2r+3)\big)M_{r-2}(\xi), r = \\
&2,3,\dots,\nu, j = 0,1,\dots,\nu \qquad (23)
\end{aligned}$$

To increase the speed of calculating the integral (22), the recursive equation (23) is used, in which $M_r(\xi)$is obtained using the following two equations:

$$M_r(\xi) = \int_0^{1-\xi} P_r\,[x + \xi]dx$$

$$M_0(\xi) = 1 - \xi, \qquad (2r+1)M_r(\xi) = P_{r+1}[\xi] - P_{r-1}[\xi]$$

The initial values for this recursive relationship are calculated directly, using the method of part-by-part integration, as follows:

$$I_0(\xi) = 2C_j\{\ln(h_j C_j)\}$$

$$I_1(\xi) = \frac{1}{2}C_j\{(1+\xi)[2\,ln(h_j C_j) - 1] - 2\xi\}$$

$$I_2(\xi) = \frac{1}{6}C_j\{6\xi(\xi+1)\,ln(h_j C_j) - (11\xi^2 + 5\xi - 4)\}$$

$$\begin{aligned}
I_3(\xi) = \frac{1}{48}C_j\{&12\xi(\xi+1)(5\xi^2 - 1)\,ln(h_j C_j) \\
&- (125\xi^3 + 65\xi^2 - 73\xi - 21)\}
\end{aligned}$$

By inserting the expression $Ir(cos\ \frac{j\pi}{\nu})$ into (21), also using (19) and then rearranging it, the following relation is obtained:
$\left[\int_{t_i}^{t_{i,j}} \ln(t_{ij}-t) y_{i+1}(t)\, dt\right] = \frac{1}{2} h_i B[y_i+1j]$ (24)

Where $y_{i+1,j} = y_{i+1}(t_{i.j})$ also, the elements of matrix $B$are obtained as follows:

$$b_{jl} = R_l \sum_{r=0}^{\nu} (-1)^r (2r+1) P_r \left[-\cos\left(\frac{l\pi}{\nu}\right)\right] Ir\left(\cos\left(\frac{j\pi}{\nu}\right)\right),\quad j=0,1,\dots,\nu,$$
$$l = 0,1,\dots,\nu$$

The continuation of the integral approximation (24) is used to obtain the numerical solution of the integral equation (1). Considering $t = (t_{ij}) \in [t_i, t_{i+1}]$ in (1), the following relation is obtained:
$y(t_{ij}) = g(t_{ij}) + \int_0^{t_{ij}} \ln(t_{ij}-s)\ f(t_{ij}, s, y(s), y(s))ds = g(t_{ij}) + \sum_{l=0}^{i-1} \int_{t_l}^{t_{l+1}} \ln(t_{ij}-s) f(t_{ij}, s, y(s), y(s))ds + \int_0^{t_{ij}} ln(t_{ij} - s)\ f(t_{ij}, s, y(s), y(s))ds,\quad i = 0,1,\dots N-1, j = 1,2,\dots,\nu$ (25)

Then, by putting (24) into (25), the numerical solution of the integral equation will be as follows:
$y_{i+1,j} = g(t_{ij}) + \sum_{l=0}^{i-1} hl \sum_{k=0}^{\nu} C_{\nu k} \ln(t_{ij} - t_{lk}) f(t_{i,j}, t_{lk}, y_{l+1,k}) + \frac{1}{2} h_i \sum_{k=0}^{\nu} b_{jk} f(t_{ij}, t_{ik}, y_{+1,k}),$ (26)
In which;

$$C_{\nu k} = R_k, y_{1,0} = g(0), \qquad y_{i+1,0} = y_{i,\nu},\quad i = 0,1,\dots\nu$$

In equation (26), to calculate the unknown ν $y_{i+1,j}$in each subinterval $[t_i, t_{i+1}](i = 0,1 \dots, N-1)$, Gauss' elimination method is used for linear equations, and Newton's iterative method is used for nonlinear equations.

4- Convergence order of the method

In this section, the convergence order of the method for the approximate solution $y_{i+1,j}$is presented. These results are formulated for $T \le e$ and the linear cases of equation (1). To examine the convergence order of the method, consider the following equation:
$y(t) = g(t) + \int_0^t \ln(t-s) F(t,s)\,(s)\,ds,$ (27)
Equation (27) for $t \in [t_i, t_{i+1}]$is rewritten in the following form:
$y(t) = g(t) + \sum_{t_l}^{t_{l+1}} \ln(t-s) F(t,s)\, y(s) ds + \int_{t_i}^{t} ln(t-s) F(t,s)\, y(s) ds,$
(28)
And by putting $t = t_{ij} \in [t_i, t_{i+1}]$, equation (28) is obtained as follows:

$y(t_{ij}) = g(t_{ij}) + \sum_{t_l}^{t_{l+1}} \ln(t_{ij} - s)F(t_{ij}, s)\, y(s)ds + \int_{t_i}^{t_{ij}} ln(t_{ij} - s)F(t_{ij}, s)\, y(s)ds,$ *(29)*

The error of the method is calculated by considering equation (29) along with the numerical approximation y_{i+1} and the exact answer y as follows:

$$\begin{aligned} e(t_{ij}) &= y(t_{ij}) - y_{i+1}(t_{ij}) \\ &= \sum_{l=0}^{i-1} \ln(t_{ij} - s)F(t_{ij}, s)\, e_l(s)ds \\ &+ \int_{t_i}^{t_{ij}} \ln(t_{ij} - s)F(t_{ij}, s)\, y(s)ds \\ &- \int_{t_i}^{t_{ij}} ln(t_{ij} - s)F(t_{ij}, s) y_{i+1}(s)ds, \end{aligned}$$

Also, since $T \le e$, then:

$$ln(t_{ij} - s) \le 1, \qquad \forall_s \in [t_l, t_{l+1}], (l = 0,1, \dots i - 1)$$

And considering the change of variable $s = t_l + vhl$, the error is simplified as follows:

$$\begin{aligned} e(t_{ij}) \le \sum_{l=0}^{i-1} hl \int_0^1 F(t_{ij}, t_l + vh_l) e_l(t_l + vh_l)dv \\ + \int_{t_i}^{t_{ij}} ln(t_{ij} - s)F(t_{ij}, s)\, y(s)ds \\ - \int_{t_i}^{t_{ij}} ln(t_{ij} - s)F(t_{ij}, s) y_{i+1}(s)ds, \end{aligned}$$

So that,

$$e(t_{ij}) = y(t_{ij}) - y_{i+1}(t_{ij}), \qquad i = 0,1, \dots, N - 1, j = 0,1, \dots, i - 1,$$

&

$$e_l(t_l + vh_l) = y(t_l + vh_l) - y_{i+1}(t_l + vh_l), \qquad l = 0,1, \dots., i - 1$$

If , $S = \{(t, s): 0 < s \le t \le T\}$ and $K_0 = \max\{|F(t, s)|: (t, s \in S)\}$.

Then:

$e(t_{ij}) \le K_0 \sum_{l=0}^{i-1} h_l \int_0^1 |e_l(t_l + vh_l)|\, dv + K_0 |q_{ij}|,$

(30)

In which,

$$q_{ij} = \int_{t_i}^{t_{ij}} \ln(t_{ij} - s)y(s)ds - \int_{t_i}^{t_{ij}} ln(t_{ij} - s)y_{i+1}(s)ds$$

Theorem 1. Consider equation (27) with the following conditions:

$$g \in C^{\nu+1}(I), F \in c^{\nu+1}(S), \qquad y^{(m)}(t_i) = y_{i+1}^{(m)}(t_i)(\forall_m, m \geq 0)$$

Then for each regular network we have:

$$\|e(t_{ij})\| = O\left(N^{-(2+\nu)}\right)$$

So that . $\|e(t_{ij})\| \coloneqq \max\{0 \leq i \leq N-1,\ \ 0 \leq j \leq \nu\}$

To prove this theorem, the following lemmas are needed.[12]

Lemma 1. Suppose that $x_i(i = 0,1, \dots N)$is a sequence of real numbers, and we also have the following relation:

$$|x_i| \leq hM \sum_{j=0}^{i-1} |x_j| + \delta\,, \quad i = 1, \dots, N,$$

Where $\delta > 0$and $M\ > 0$, which is usually independent of h, then:

$$|x_i| \leq (hM|x_0| + \delta)e^{M\ ih}, \quad i = 1, ..$$

Lemma 2. For Chebyshev polynomials, the following statement holds:

$$\sum\nolimits_{l=0}^{\nu}{}'' \, C_l^j\, T_r\left[-\cos\left(\frac{l\pi}{\nu}\right)\right] = \begin{cases} \frac{\nu}{2^{2j}}\binom{2j}{j-r}(1+\delta) \\ 0 \end{cases} \qquad (31)$$

Where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Proof. To prove the lemma we consider the following two cases:

1) Suppose $j = 0$. So for $r = 0$ the following relation holds:

$$\sum_{l=0}^{\nu''} C_l^0\, T_0\left[-\cos\left(\frac{l\pi}{\nu}\right)\right] = \nu$$

Also for $r < 0$we have:

$$\sum\nolimits_{l=0}^{\nu}{}'' \, C_l^0\, T_r\left[-\cos\left(\frac{l\pi}{\nu}\right)\right] = (-1)^r \sum\nolimits_{l=0}^{\nu}{}'' \left(\frac{lr\pi}{\nu}\right)$$

(32)

Given that the following relationship holds [13]:

$$\sum_{l=0}^{\nu} \cos(lx) = \frac{\cos\left(\frac{\nu}{2}x\right)\sin\left(\left(\frac{\nu+1}{2}\right)x\right)}{sin\,\frac{x}{2}}$$

So, we have,

$$\sum_{l=0}^{\nu}\sum_{l=0}^{''} cos\,(lx) = \frac{1}{2}cot\left(\frac{x}{2}\right)\sin(\nu x),$$

Then, by putting $x = \frac{r\pi}{\nu}$, we arrive at the following expression:

$$\sum_{l=0}^{\nu\ ''} cos\left(\frac{lr\pi}{\nu}\right) = 0$$

Finally, if $r < 0$, the following result is obtained:

$$\sum_{l=0}^{\nu\ ''}\sum_{l=0}^{\nu\ ''} C_l^j T_0\left[-\cos\left(\frac{l\pi}{\nu}\right)\right] = 0$$

2) Considering $j \neq 0$, according to (33), relation (34) is obtained as follows:

$$C_l^j = \left(\frac{1}{2}\left(1 - \cos\left(\frac{lr\pi}{\nu}\right)\right)\right)^j = \ sin^{2j}\left(\frac{l\pi}{2\nu}\right) \qquad (33)$$

$$\sum_{l=0}^{\nu\ ''} C_l^j T_r\left[-\cos\left(\frac{l\pi}{\nu}\right)\right] = (-1)^r \sum_{l=0}^{\nu\ ''} \cos\left(\frac{lr\pi}{\nu}\right)\sin^{2j}\left(\frac{l\pi}{2\nu}\right)$$

Also, considering relation (35) [13], substituting it into (36) we have:

$$\sin^{2n}(x) = \frac{1}{2^{2n}}\left[\textstyle\sum_{k=0}^{n-1}(-1)^{n-k_2}\binom{2n}{k} cos(2(n-k)x) + \binom{2n}{k}\right]$$

(35)

$$\sum_{l=0}^{\nu\ ''} C_l^j T_r\left[-\cos\left(\frac{l\pi}{\nu}\right)\right]$$

$$= (-1)^r \sum_{l=0}^{\nu\ ''} \frac{1}{2^{2j}} \sum_{k=0}^{2j} (-1)^{j-k} \cos\left(\frac{lr\pi}{\nu}\right)\cos\left(\frac{(j-k)l\pi}{\nu}\right),$$

And using (32), the following expression is obtained:

$$\sum_{l=0}^{\nu\ ''} C_l^j T_r\left[-\cos\left(\frac{l\pi}{\nu}\right)\right] = \begin{cases} \frac{\nu}{2^{2j}}\binom{2j}{j-r}(1+\delta_{r\nu}) & if\ r \leq j \\ 0 & if\ r \leq j \end{cases}$$

Lemma 3. Consider equation (27) with the following conditions:

$$g \in C^{\nu+1}(I), F \in C^{\nu+1}(S), \qquad y^{(m)}(t_i) = y_{i+1}^{(m)}(t_i)(\forall_m, m \geq)$$

Then for any regular network we have:

$$\|q_{ij}\| \leq Ah^{2+\nu}$$

So,

A is a general constant dependent on $\{C_j\}$.

Proof. The proof of this lemma for Heichbischoff polynomials is given below, it is proved similarly for Legendre polynomials. For more details, see [1-2]. According to relation (31), we have:

$$q_{ij} = \int_{t_i}^{t_{ij}} ln(t_{ij} - s)y\,(s)ds - \int_{t_i}^{t_{ij}} ln(t_{ij} - s)y_{i+1}(s)ds$$

$$= \int_{t_i}^{t_{ij}} ln(t_{ij} - s)y(s)ds - h_i \sum_{k=0}^{\nu} b_{jk}\, y_{i+1}\,,k$$

$$h_i \left\{\int_0^{C_j} ln(h_i\theta)\, y(t_i + (C_i - \theta)h_i)d\theta - \sum_{k=0}^{\nu} b_{jk} y_{i+1}(t_i + C_k h_i)\right\} \quad (36)$$

Using Taylor series expansion, the following expression is obtained:

$$q_{ij} = h_i \sum_{m=0}^{\nu+1} \frac{h_i^m}{m!} y^{(m)}(t_i)\left[\int_0^{C_j} \ln(h_i\theta)(C_j - \theta)^m d\theta - \sum_{k=0}^{\nu} b_{jk} C_k^m\right]$$

Where $\Delta(m)$is simplified as follows:

$$\Delta(m) = \int_0^{C_j} \ln(h_i\theta)(C_j - \theta)^m\, d\theta$$

$$= C_j^{m+1} \sum_{n=0}^{m} \frac{(-1)^n}{(n+1)^2}\left[n + 1\, ln(h_i C_j) - 1\right]$$

In this section, we are looking for m that satisfies the following condition:

$$\sum_{k=0}^{\nu} b_{jk} C_k^m \neq m$$

If the value of m is equal to p, then we have:

$$|q_{ij}| \leq Ah^{1+\nu+p},$$

Where $A$is a general constant dependent on C_j. According to Lemma (3), we consider the following two cases:

3) Suppose m ≤ ν

$$\sum_{k=0}^{\nu} b_{jk} C_k^m = \frac{1}{2^{2m}} \int_0^{1-\xi} \ln\left(\frac{h_i}{2}x\right) \sum_{r=0}^{m} \frac{1}{1+\delta_{r0}+\delta_{m\nu}\delta_{r\nu}} \binom{2m}{m-r} (1 + \delta_{r\nu}) T_r[-(x+\xi)]dx = \frac{1}{2^{m+1}} \sum_{n=0}^{m} \binom{m}{n} (-1)^n (1 - \xi)^{m-n} \int_0^{1-\xi} x^n \ln\left(\frac{h_i}{2}x\right) dx. \quad (37)$$

Therefore, the following result is obtained:

$$\sum_{k=0}^{\nu} b_{jk} C_k^m = \Delta(m) \qquad m \leq \nu \qquad (38)$$

The exact expression of this equation is $y(t) = t$.

Table (1) is the numerical results for example (1) in which the maximum absolute error for the finite expansion method of Chebyshev polynomials [2] is presented at the point $t = 1$and for different values. Also, according to the table, it can be said that with increasing ν and decreasing h, the maximum absolute error decreases and the presented results become better.

Table 1: Maximum absolute error for $(\nu = 3,4,5)$ for the finite expansion method of Chebyshev polynomials.

$\nu = 5$	$\nu = 4$	$\nu = 3$	$h\backslash\nu$
$0/83078D - 04$	$0.59118D - 04$	$0/49394D - 03$	$\frac{1}{2}$
$0/71656D - 05$	$0/46784D - 05$	$0/16847 - 03$	$\frac{1}{4}$
$0/27329D - 05$	$0/10793D - 05$	$0/51037D - 04$	$\frac{1}{8}$
$0/83359D - 07$	$0/67860D - 06$	$0/14731D - 04$	$\frac{1}{16}$
$0/78273D - 07$	$0/23766D - 06$	$0/40237D - 05$	$\frac{1}{32}$
$0/25873D - 07$	$0/68281D - 07$	$0/10574D - 05$	$\frac{1}{64}$
$0/71673D - 08$	$0/18265D - 07$	$0/27183D - 06$	$\frac{1}{128}$
$0/18822D - 08$	$0/47447D - 08$	$0/69044D - 07$	$\frac{1}{256}$
$0/48485D - 09$	$0/12144D - 08$	$0/17519D - 07$	$\frac{1}{512}$
$0/10312D - 09$	$0/31210D - 09$	$0/44020D - 08$	$\frac{1}{1024}$

Also, Table (2) is related to Example (1) for the method of finite expansion of Legendre polynomials [1], in which the maximum absolute error is shown at the point $t = 1$and for different ν.

Table 2: Maximum absolute error for $(\nu = 3,4,5,6)$.

$h\backslash\nu$	$\nu = 3$	$\nu = 4$	$\nu = 5$	$\nu = 6$
$\frac{1}{2}$	$0/32865D - 03$	$0/46950D - 04$	$0/73077D - 04$	$0/48002 - 05$
$\frac{1}{4}$	$0/1224503 - 03$	$0/17024D - 05$	$0/20409D - 05$	$0/16083D - 05$

$\frac{1}{8}$	$0/38621D-04$	$0/47293D-05$	$0/53655D-05$	$0/43715D-06$
$\frac{1}{16}$	$0/11323D-04$	$0.412264D-05$	$0/13707D-05$	$0/11286D-06$
$\frac{1}{32}$	$0/32393D-05$	$0/31102D-06$	$0/34577D-06$	$0/28619D-07$
$\frac{1}{64}$	$0/91395D-06$	$0.78238D-07$	$0/86761D-07$	$0/72046D-08$
$\frac{1}{128}$	$0/91395D-06$	$0.78238D-07$	$0/86761D-07$	$0/18093D-08$
$\frac{1}{256}$	$0/70518D-07$	$0.49166-07$	$0/54399D-08$	$0/45627D-09$
$\frac{1}{512}$	$0/19331D-07$	$0/12337D-08$	$0/13588D-08$	$0/11646D-09$
$\frac{1}{1024}$	$0/52995D-08$	$0/31536D-09$	$0/33972D-09$	$0/38367-10$

For the finite expansion method of Legendre polynomials, we can compare these two methods with the co-localization method with continuous piecewise polynomials of Braner [13] - for example (1). In Table (3) the maximum absolute error for $\nu = 2$at the point $t = 1$, for all three methods, is presented.

Table 3: Investigation of the maximum absolute error for the finite expansion methods of Chebyshev and Legendre polynomials and Brauner's co-localization method for $\nu = 2$.

h	**Chebyshev expansion method [2]**	**Legendre expansion method [1]**	**Brauner's colocalization method [6]**
$\frac{1}{2}$	$0/11224D-02$	$0/10051D-02$	$0/15328-01$
$\frac{1}{4}$	$0/55565D-03$	$0/56226D-03$	$0/85413D-01$
$\frac{1}{8}$	$0/17102D-03$	$0/17102D-03$	$0/21196D-01$
$\frac{1}{16}$	$0/46024D-04$	$0/46179D-04$	$0/10871D-02$

$\frac{1}{32}$	$0.11817D-04$	$0/11836D-04$	$0/78353D-02$
$\frac{1}{64}$	$0.29828-05$	$0/29851D-05$	$0/55820D-02$
$\frac{1}{128}$	$0.74527-06$	$0/74827D-06$	$0/39856D-02$
$\frac{1}{256}$	$0/118729D-06$	$0/18732D-06$	$0/280230D-02$
$\frac{1}{512}$	$0/46835D-07$	$0/46839D-07$	$0/19842-02$
$\frac{1}{1024}$	$0/11696D-07$	$0/11696D-07$	$0/14020-02$

These results show that the Chebyshev and Legendre polynomial expansion methods have the same accuracy. It is also observed that these two methods have better accuracy than the co-localization method with continuous piecewise Brauner polynomials. Figures (1) and (2) show the error plots for the finite expansion method of Chebyshev and Legendre polynomials for $\nu=2$and $\nu=16$ on the interval [0,1], respectively.

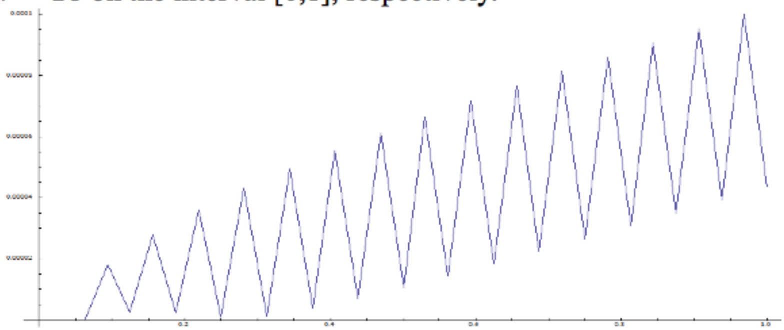

Figure 1: Error diagram for the Chebyshev expansion method for ν=2and n=16 on the interval [0,1].

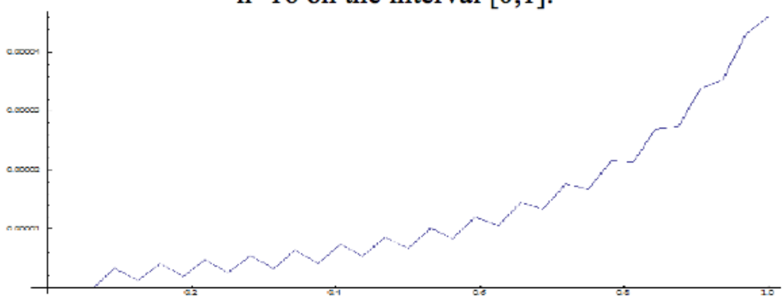

Figure 2: Error plot for the Legendre expansion method for $\nu = 2$and n=16 on the interval [0,1].

Example 2. Consider the following integral equation:

$$(t) = t + 0.25t^2 - 0.5t^2 \ln t + \int_0^t \ln(t-s)\left(t - y(s)\right)ds$$

The exact form of this equation is $y(t) \ = t$.

Table 4: Maximum absolute error for Chebyshev and Legendre expansion methods for $\nu = (2)$.

h	**Chebyshev expansion method [2]**	**Legendre expansion method [1]**
$\frac{1}{2}$	$0/343924D - 03$	$0/386706D - 03$
$\frac{1}{4}$	$0/155428D - 03$	$0/159417D - 03$
$\frac{1}{8}$	$0/489828D - 04$	$0/494768D - 04$
$\frac{1}{16}$	$0/4135896D - 04$	$0/13651D - 04$
$\frac{1}{32}$	$3/5639D - 04$	$3/57665D - 06$
$\frac{1}{64}$	$0.91385 - 07$	$9/14806D - 07$

In this example, as you can see, the accuracy for these two methods is almost the same. We will briefly explain this example using the finite expansion method of Chebyshev polynomials for $\nu = 2$and n=2. For these values, the points are as follows:

$t[0,0] = 0, t[0,1] = 0.25, t[0,2] = t[1,0] = 0.5, t[1,1] = 0.75, t[1,2] = 1,$

The real answer at these points is obtained as follows:

$$y[1,0] = 0, y[1,1] = 0.25, y[1,2] = t[2,0] = 0.5, y[1,1] = 0.75, y[2,2] = 1,$$

First, $I_{r,j}\left(\cos\left(\frac{j\pi}{\nu}\right)\right)$is obtained:

$I_{0,1} = -2/38629$ $\quad$ $I_{0,1} = -3/38629$

$I_{1,1} = -0/943147$ $\quad$ $I_{1,2} = 1$

$I_{0,1} = -1/23988$ $\quad$ $I_{2,2} = -0/68432$

Using that, the elements of matrix $B$are obtained as follows:

$b_{1,0} = -0.379089,$ $\quad$ $b_{1,1} = -0.906543,$ $\quad$ $b_{1,2} = -0/0924845$

$b_{2,0} = -0.0877468 \qquad b_{2,1} = -1.01765, \qquad b_{2,2} = -0/587747,$

Also, $C_{v,k}$ results in the following form:

$$C_{v,0} = 1/6 \qquad C_{v,1} = 2/3 \qquad C_{v,2} = 1/6$$

Given these values, the approximate value of the integral equation can be obtained by solving the following systems:

$$\begin{bmatrix} -0.546729 & -0.4624220 \\ +508827 & -\frac{0}{706127} \end{bmatrix} \begin{bmatrix} y'^{[1,1]} \\ y'^{[1,2]} \end{bmatrix} = \begin{bmatrix} \frac{0}{159803} \\ 0.225857 \end{bmatrix}$$

After solving the above system, $y'[1,1]$and $y'[1,2]$are calculated as follows:

$$y'[1,1] = 0/25 \qquad y'[1,1] = 0/5$$

Then we put:

$$y'[1,2] \ = \ y'[2,0]$$

$$\begin{bmatrix} -0.546729 & -0.4624220 \\ +508827 & -\frac{0}{706127} \end{bmatrix} \begin{bmatrix} y'^{[2,1]} \\ y'^{[2,2]} \end{bmatrix} = \begin{bmatrix} \frac{0}{36172} \\ \frac{0}{302625} \end{bmatrix}$$

By solving the above system, the results are obtained as follows:

$$y'[2,1] = \frac{0}{750334}, \qquad y'[1,1] = \frac{1}{00033}.$$

The absolute error at these points is as follows:

$$e[1,0] = 0, e[1,1] = 0, e[1,2] = 0, e[2,0] = 0, e[2,1] = \frac{0}{0003255649},$$

As can be seen, the maximum absolute error is $0/000343924$, which is also stated in Table (4). Similarly, for different v and n, the approximate value of the solution to the integral equation and the maximum absolute error are obtained.

6- Conclusion

In this thesis, the single-step ν-step method, with a numerical nature using finite expansion of orthogonal Chebyshev and Legendre polynomials, has been investigated in order to solve weak singular Volterra integral equations with logarithmic kernels.

Numerical results show a significant improvement compared to the colocal method with continuous piecewise polynomials of Brunner [6]. Also, by decreasing h and increasing v, better results are obtained. Although the computational accuracy of the method is the same for Chebyshev and Legendre polynomials, but by changing the nodal points, provided that the continuity of the kernel is preserved and the start and end points of the interval are included, we expect the accuracy of the calculations to change. In this paper, we have used Gauss Chebyshev-

Lebato points. If the weight functions of Chelishkov polynomials can be calculated at the points [0,1], we expect that these polynomials can also be used to numerically solve integral equations. Considering the Chebyshev polynomials of the second kind and their roots, as well as their accuracy in L^1space, we can expect that using these polynomials will yield better results. This method can be used for various types of integral equations, with uniform and non-uniform grids.

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