2 – Assist. prof. Dr.Abdulateef Hassan Shoman and Lecturer Dr.Ali Hassan Zayer Some reliable techniques for solving higher-orderVolterra integro-differential equations

Khawlah Hashim Hussain

Abstract

1. In this paper, the Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM) are applied to solve boundary value problems for higher-order Volterra integro-differential equations. The numerical results obtained with minimum amount of computation are compared with the exact solutions to show the efficiency of the techniques. The results show that the variational iteration method is of high accuracy, more convenient and efficient for solving integro-differential equations.

Keywords: Adomian decomposition method, variational iteration method, integro-differential equation.

2. Introduction In this work, we consider the Volterra integro-differential equation of the second kind as follows:

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \int_a^x K(x,t)G(u(t))dt,$$

:with the initial or boundary conditions	
,(r = 0,1,2,…,(n − 1 ,$u^{(r)}(a) = \alpha_r$	(2)
,(r = n,(n + 1),…,(k − 1 ,$u^{(r)}(b) = \beta_r$	(3)

where $u^{(j)}(x)$ is the j^{th} derivative of the unknown function $u(x)$ that will be determined, $K(x,t)$ is the kernel of the equation, $a \le x \le b$, $f(x)$ and $p_j(x)$ are analytic functions, G is analytic function of u, and α_r; $0 \le r \le (n-1)$ and β_r; $n \le r \le (k-1)$ are real finite constants.

In recent years there has been a growing interest in the integro-differential equation. The integro-differential equations be an important branch of modern mathematics. It arises frequently in many applied areas which include engineering, electrostatics, mechanics, the theory of elasticity, potential, and mathematical physics [3,4,6,10,27].

Recently, Wazwaz (2001) presented an efficient and numerical procedure for solving boundary value problems for higher-order integro-differential equations. A variety of methods, exact, approximate and purely numerical techniques are available to solve nonlinear integro-differential equations. These methods have been of great interest to several authors and used to solve many nonlinear problems. Some of these techniques are Adomian decomposition method [4,24], modified Adomian decomposition method [30], Variational iteration method [7,32,33] and homotopy perturbation method [30] and many methods for solving integrodifferential equations [2,3,6,16-21,28].

More details about the sources where these equations arise can be found in physics, biology, and engineering applications as well as in advanced integral equations. Some works based on an iterative scheme have been focusing on the development of more advanced and efficient methods for integro-differential equations such as the variational iteration method which is a simple and Adomian decomposition method [8,9,24,31], and the modified decomposition method for solving Volterra-Fredholm integral and integro-differential equations which is a simple and powerful method for solving a wide class of nonlinear problems [24]. The Taylor polynomial solution of integro-differential equations has been studied in [27]. The use of Lagrange interpolation in solving integro-differential equations was investigated by Marzban [26]. The VIM has been successfully applied for solving integral and integro-differential equations [5,24,30].

A variety of powerful methods has been presented, such as the homotopy analysis method [30], homotopy perturbation method [6,30], operational matrix with Block-Pulse functions method [3], variational iteration method [5,30] and the Adomian decomposition method [4,24,30]. Some fundamental works on various aspects of modifications of the Adomian's decomposition method are given by Araghi [1]. The modified form of Laplace decomposition method has been introduced by Manafianheris [25]. Babolian et. al, [3], applied the new direct method to solve nonlinear Volterra-Fredholm integral and integro-differential equation using operational matrix with block-pulse functions. The Laplace transform method with the Adomian decomposition method to establish exact solutions or approximations of the nonlinear Volterra integro-differential equations, Wazwaz [31]. Recently, the authors have used several methods for the numerical or the analytical solutions of linear and nonlinear Volterra and Fredholm integro-differential equations

[11–15,22–24,30].

In this work, our aim is to solve a general form of boundary value problems for higher–order Volterra integro–differential equations by using the ADM and VIM.

3. Volterra Integro–differential Equation of Second Kind We consider the Volterra integro–differential equation of the second kind as follows:

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \int_a^x K(x,t)G(u(t))dt,$$

(4)

We can rewrite Eq.(4) as follows:

$$p_j(x)u^{(j)}(x) = f(x) + \int_a^x K(x,t)G(u(t))dt,$$

$$u^{(k)}(x) = \frac{f(x)}{p_k(x)} + \int_a^x \frac{K(x,t)G(u(t))}{p_k(t)}dt - \sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)}u^{(j)}(x)$$

$$u^{(k)}(x) = \frac{f(x)}{p_k(x)} + \int_a^x \frac{K(x,t)G(u(t))}{p_k(t)}dt - \sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)}u^{(j)}(x)$$

(5)

Let us set L^{-1} is the multiple integration operator as follows:

$$L^{-1}(.) := \int_a^x \int_a^x \dots \int_a^x \int_a^x (.)\, \underbrace{dtdt \dots dtdt}$$

(6)

k–times

From Eq.(5) and Eq.(6)

$$u(x) = L^{-1}\{\frac{f(x)}{p_k(x)}\} + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!}\alpha_r + L^{-1}\{\int_a^x \frac{K(x,t)G(u(t))}{p_k(t)}dt\}$$
$$-L^{-1}\{\sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} u^{(j)}(x)\}.$$

(7)

We can obtain the term $\sum_{r=0}^{k-1} \frac{(x-a)^r}{r!}\alpha_r$ from the initial conditions. From [8], we have

(8)

$$L^{-1}\{\int_a^x \frac{K(x,t)G(u(t))}{p_k(t)}dt\} = \int_a^x \frac{(x-t)^k K(x,t)G(u(t))}{(k!) \quad p_k(t)}dt,$$

also

$$L^{-1}\{\sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} u^{(j)}(x)\} = \sum_{j=0}^{k-1} \int_a^x \frac{(x-t)^{k-1} p_j(t)}{(k-1)! \quad p_k(t)} u^{(j)}(t)dt$$

(9)

By substituting Eq.(8)and Eq.(9) in Eq.(7) we obtain

$$u(x) = L^{-1}\{\frac{f(x)}{p_k(x)}\} + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!}\alpha_r + \int_a^x \frac{(x-t)^k K(x,t)G(u(t))}{(k!) \quad p_k(t)}dt$$
$$-\sum_{j=0}^{k-1} \int_a^x \frac{(x-t)^{k-1} p_j(t)}{(k-1)! \quad p_k(t)} u^{(j)}(t)dt.$$

(10)

We set,

$$L^{-1}\{\frac{f(x)}{p_k(x)}\} + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!}\alpha_r = F(x)$$

$$\frac{(x-t)^k K(x,t)}{(k!)\quad p_k(t)} = K_1(x,t)$$

$$\frac{(x-t)^{k-1} p_j(x)}{(k-1)!\quad p_k(x)} = K_2(x,t).$$

So, we have one−dimensional integro−differential equation as follows:

$$u(x) = F(x) + \int_a^x K_1(x,t)G(u(t))dt - \sum_{j=0}^{k-1}\int_a^x K_2(x,t)u^{(j)}(t)dt.$$

Description of the Methods

In this section, we present the semi−analytical techniques based on ADM and VIM to solve Volterra integro−differential equation:

3.1 Adomian Decomposition Method (ADM)

The ADM is applied to the following general nonlinear equation [1,30]:

$$Lu + Ru + Nu = f(x), \quad (11)$$

where u is the unknown function, L is the highest−order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L and Nu represents the nonlinear terms and f is the source term. Applying the inverse operator L^{-1} to both sides of Eq.(11) and using the given conditions we obtain

$$u = F(x) - L^{-1}(Ru) - L^{-1}(Nu), \quad (12)$$

where the function F(x) represents the terms arising from integrating the source term f(x). The nonlinear operator Nu = G(u) is decomposed as

$$G(u) = \sum_{n=0}^{\infty} A_n \quad , \quad (13)$$

where A_n; n >0 are the Adomian polynomials determined formally as fol− lows:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[G\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]\right]_{\lambda=0}.$$

The Adomian polynomials were introduced in [30] as:

$$A_0 = G(u_0); \quad A_1 = u_1 G'(u_0); \quad A_2 = u_2 G'(u_0) + \frac{1}{2} u_1^2 G''(u_0)$$

$$A_3 = u_3 G'(u_0) + u_1 u_2 G''(u_0) + \frac{1}{3} u_1^3 G'''(u_0), \ldots$$

The standard decomposition technique represents the solution of u in Eq.(11) as the following series:

$$u = \sum_{n=0}^{\infty} u_n$$

(14)

The nonlinear terms $G(u(t))$ $D^j(u(x)), (D^j = \frac{\partial^j}{\partial x^j}$ and is derivative operator), are usually represented by an infinite series of the so called Adomian polynomials as follows:

$$G(u(t)) = \sum_{i=0}^{\infty}, \sum_{i=0}^{\infty} A_i, D^j(u(x)) = \sum_{i=0}^{\infty} Bij, \sum_{i=0}^{\infty} Bij,$$

where A_i and B_{ij} ($i \geq 0, j = 0, 1, \ldots, k - 1$) are the Adomian polynomials. where, the components $u_0, u_1, \ldots$ are usually determined recursively by

$u_0 = F(x)$, (15)

$un+1 = -L^{-1}(Run) - L^{-1}(An)$.

Substituting (12) into (15) leads to the determination of the components of u. Having determined the components $u_1, u_2, \ldots$, the solution u in a series form defined by Eq.(14).

Now, we apply ADM to find the approximate solutions of Eq.(11), we can write the iterative formula as follows:

$u_0(x) = F(x)$,

(16)

$u_{n+1}(x) =$

$$\int_a^x k_1(x,t) A_n dt - \sum_{j=0}^{K-1} \int_a^x k_1(x,t) A_n dt$$

$\int_a^x k_1(x,t)A_n dt -$

$$\sum_{j=0}^{K-1} \int_a^x k_1(x,t)A_n dt$$

Convergence aspects of the ADM have been investigated in [19]. For later numerical computation, let the expression

$$Y_n(x) = \sum_{k=0}^{n-1} u_k \sum_{k=0}^{n-1} u_k$$

denote the n–term approximation to u(x).

3.2 Variational Iteration Method (VIM)

This method has been applied to solve a large class of linear and nonlinear problems with approximations converging rapidly to exact solutions.

The main idea of this method is to construct a correction functional form using general Lagrange multipliers. These multipliers should be chosen such that its correction solution is superior to its initial approximation, called trial function. It is the best within the flexibility of trial functions. Accordingly, Lagrange multipliers can be identified by the variational theory [29,30]. A complete review of VIM is available in [7,16].

The initial approximation can be freely chosen with possible unknowns, which can be determined by imposing boundary/initial conditions. To illustrate, we consider the following general differential equation:

$$Lu(t) + Nu(t) = f(t), \tag{17}$$

where L is a linear operator, N is a nonlinear operator and f(t) is inhomogeneous term. According to variational iteration method [5,24], the terms of a sequence u_n are constructed such that this sequence converges to the exact solution. The terms u_n are calculated by a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)(Lu_n(\tau) + N\tilde{u}(\tau) - f(\tau))d\tau. \tag{18}$$

The successive approximation $u_n(t), n \geq 0$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0.

The zeroth approximation u_0 may be selected using any function that just satisfies at least the initial and boundary conditions with λ determined, sev–

eral approximations $u_n(t), n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(t) \quad u(t)=\lim_{n\to\infty} u_n(t) \qquad (19)$$

4. Numerical Results

In this section, we present the numerical techniques based on ADM and VIM to solve Volterra integro-differential equations:

Example 1.

Consider the Volterra integro-differential equation as follow:

$$u^{(4)}(x) - u(x) = x(1+e^x) + 3e^x - \int_0^x u(t)dt, \quad 0 < x < 1$$

(20)

with the boundary conditions:

$$u(0) = 1,\ u(1) = 1 + e,\ u''(0) = 2,\ u''(1) = 3e. \qquad (21)$$

The exact solution is $u(x) = 1 + xe^x$.

The recursive of ADM

$$u_0 = 1 + Ax + x^2 + \frac{1}{3!}Bx^3 + L^{-1}(x + xe^x + 3e^x)$$

$$u_{n+1} = L^{-1}\{u_n(x)\} - L^{-1}\{\int_0^x A_n(t)dt\},\ n \geq 0.$$

, (22)

Using the recursive algorithm,

$$\Upsilon_2(x) = (f_0 + \frac{1}{360}x^6) + (f_1 - \frac{1}{1209600}x^{10} - \frac{1}{19958400}x^{11})$$

$$\Upsilon_3(x) = (f_0 + \frac{1}{144}x^6 + \frac{1}{1680}x^7) + (f_1 - \frac{1}{19958400}x^{11} - \frac{1}{159667200}x^{12})$$

$$\Upsilon_4(x) = (f_0 + \frac{1}{144}x^6 + \frac{1}{840}x^7 + \frac{1}{10080}x^8) + (f_1 + \frac{1}{39916800}x^{11} - \frac{1}{239500800}x^{12} - \frac{1}{1556755200}x^{13})$$

where

$$f_0 = 1 + Ax + x^2 + \frac{1}{3!}Bx^3 + \frac{1}{8}x^4 + \frac{1}{4!}x^5,$$
$$f_1 = \frac{1}{4!}x^4 + \frac{A-1}{120}x^5 + \frac{2-A}{720}x^6 + \frac{B-2}{5040}x^7 + \frac{3-B}{40320}x^8 + \frac{1}{181440}x^9$$

,

where the constants A = u'(0) and B = u'''(0) can be determined by im– posing the boundary condition (21) in $u_1(x)$, then we have

A = 0.9740953834, B = 3.1897820557 for Y_2

A = 0.9944789128, B = 3.0382130987 for Y_3

A = 0.9989095252, B = 3.0073098797' for Y_4

The rest of components of the iteration formulas can be obtained in the same manner.

Using VIM, the iterative formula can be written in the following form:

$$u_{n+1}(x) \tag{23}$$
$$= u_n(x) + \int_0^x \left(\frac{1}{3!}s^3 - \frac{x}{2!}s^2 + \frac{x^2}{2!}s - \frac{1}{3!}x^3\right)\left[u_n^{(4)}(s) - u_n(s) - s(1+e^s) - 3e^s + \int_0^s u_n(t)dt\right]ds$$

Now, starting with the initial solution

$$u_0(x) = 1 + Ax + \frac{1}{2}x^2 + \frac{1}{3!}Bx^3,$$
$$u_1(x) = 1 + Ax + x^2 + \frac{1}{3!}Bx^3 + 0.042(24 + 8.882(10^{-16})e^x(-1+x)(2.702(10^{16} + x(3+x))))$$
$$+x^2(-12 + x(-8 + x(1 + A(0.2 - 0.3)x + x2(0.07 + x(-0.01 + 0.005B - 0.0006Bx)))))$$

where the constants A = u'(0) and B = u'''(0) can be determined by impos– ing the boundary condition (21) in $u_1(x)$, then we have

A = 0.9991323308076788,

B = 3.005849664438371.

The rest of components of the iteration formulas can be obtained in the same manner.Conclusion

We discussed the ADM and VIM to solve boundary value problems for higherorder integro–differential equations. The results obtained with the minimum amount of computation are compared with the exact solutions to show the efficiency of the techniques. The results show that the variational iteration

method is of high accuracy, more convenient and efficient for solving integro-differential equations.

References

1. M.A. Araghi, S.S. Behzadi, Solving nonlinear Volterra-Fredholm integro-differential equations using the modified Adomian decomposition method, Comput. Methods in Appl. Math. 9 (2009), 321-331.
2. S.H. Behiry, S.I. Mohamed, Solving high-order nonlinear Volterra-Fredholm integro-differential equations by differential transform method, Natural Science, 4 (2012), no. 8, 581-587.
3. E. Babolian, Z. Masouri, S. Hatamzadeh, New direct method to solve nonlinear Volterra-Fredholm integral and integro differential equation using operational matrix with Block-Pulse functions, Progress in Electromagnetic Research, B 8 (2008), 59–76.
4. S.M. El-Sayed, D. Kaya, S. Zarea, The decomposition method applied to solve highorder linear Volterra-Fredholm integro-differential equations, International Journal of Nonlinear Sciences and Numerical Simulation, 5 (2004), no. 2, 105-112.
5. F.S. Fadhel, A.O. Mezaal, S.H. Salih, Approximate solution of the linear mixed Volterra-Fredholm integro-differential equations of second kind by using variational iteration method, Al- Mustansiriyah, J. Sci. 24 (2013), no. 5, 137-146.
6. M. Ghasemi, M. kajani, E. Babolian, Application of He's homotopy perturbation method to nonlinear integro differential equations, Appl. Math. Comput. 188 (2007), 538–548.
7. J.H. He, S.Q. Wang, Variational iteration method for solving integro-differential equations, Phys. Lett. A, 367 (2007), 188-191.
8. A.A. Hamoud, K.P. Ghadle, Existence and uniqueness of the solution for VolterraFredholm integro-differential equations, Journal of Siberian Federal University. Mathematics & Physics, 11 (2018), no. 6, 692–701.
9. A.A. Hamoud, K.P. Ghadle, Modified Laplace decomposition method for fractional Volterra-Fredholm integro-differential equations, Journal of Mathematical Modeling, 6 (2018), no. 1, 91-104.
10. A.A. Hamoud, K.P. Ghadle, Homotopy analysis method for the first order fuzzy Volterra-Fredholm integro-differential equations, Indonesian J. Elec. Eng. & Comp. Sci. 11 (2018), no. 3, 857–867.
11. A.A. Hamoud, K.P. Ghadle, Some new existence, uniqueness and convergence results for fractional Volterra-Fredholm integro-differential equations, J. Appl. Comp. Mech. 5 (2019), no. 1, 58–69.
12. A.A. Hamoud, K.P. Ghadle, Existence and uniqueness of solutions for fractional mixed Volterra-Fredholm integro-differential equations, Indian J. Math. 60 (2018), no. 3, 375–395.

13. A.A. Hamoud, K.P. Ghadle, The approximate solutions of fractional VolterraFredholm integro−differential equations by using analytical techniques, Probl. Anal. Issues Anal. 7(25) (2018), no. 1, 41–58.

14. A.A. Hamoud, K.P. Ghadle, S.M. Atshan, The approximate solutions of fractional inte−gro−differential equations by using modified Adomian decomposition method, Khayyam Journal of Mathematics, 5 (2019), no. 1, 21–39.

15. A.A. Hamoud, K.P. Ghadle, The reliable modified of Laplace Adomian decomposition method to solve nonlinear interval Volterra-Fredholm integral equations, Korean J. Math. 25 (2017), no. 3, 323–334.

16. A.A. Hamoud, L.A. Dawood, K.P. Ghadle, S.M. Atshan, Usage of the modified variational iteration technique for solving Fredholm integro-differential equations, International Journal of Mechanical and Production Engineering Research and De−velopment , 9 (2019), no. 2, 895–902.

17. A.A. Hamoud, K.H. Hussain, K.P. Ghadle, The reliable modified Laplace Adomian decomposition method to solve fractional Volterra-Fredholm integro-differential equations, Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms, 26 (2019), 171–184.

18. A.A. Hamoud, K.P. Ghadle, Modified Adomian decomposition method for solving fuzzy Volterra-Fredholm integral equations, J. Indian Math. Soc. 85 (2018), no. 1−2, 52–69.

19. A.A. Hamoud, K.P. Ghadle, M. Bani Issa, Giniswamy, Existence and uniqueness theorems for fractional Volterra-Fredholm integro-differential equations, Int. J. Appl. Math. 31 (2018), no. 3, 333–348.

20. A.A. Hamoud, A.D. Azeez, K.P. Ghadle, A study of some iterative methods for solving fuzzy Volterra-Fredholm integral equations, Indonesian J. Elec. Eng. & Comp. Sci. 11 (2018), no. 3, 1228–1235.

21. A.A. Hamoud, K.P. Ghadle, Usage of the homotopy analysis method for solving fractional Volterra-Fredholm integro-differential equation of the second kind, Tam−kang Journal of Mathematics, 49 (2018), no. 4, 301–315.

22. A.A. Hamoud, M. Bani Issa, K.P. Ghadle, M. Abdulghani, Existence and convergence results for caputo fractional Volterra integro-differential equations, Journal of Mathematics and Applications, 41 (2018), 109–122.

23. A.A. Hamoud, M. Bani Issa, K.P. Ghadle, Existence and uniqueness results for nonlinear Volterra-Fredholm integro-differential equations, Nonlinear Functional Analysis and Applications, 23 (2018), no. 4, 797–805.

24. A.M. Jerri, Introduction to Integral Equations with Applications. New York, Wiley, 1999.

25. J. Manafianheris, Solving the integro−differential equations using the modified Laplace Adomian decomposition method, Journal of Mathematical Extension, 6 (2012), no. 1, 41–55.

26. H.R. Marzban, S.M. Hoseini, Solution of nonlinear Volterra-Fredholm integrodifferential equations via hybrid of Block-Pulse functions and Lagrange interpolating polynomials, Hindawi Publishing Corporation, Advances in Numerical Analysis, 868 (2012), no. 279, 1–14.

27. Y. Salih, S. Mehmet, The approximate solution of higher order linear Volterra-Fredholm integro-differential equations in term of Taylor polynomials, Appl. Math.